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A CLASS OF NONPARAMETRIC TESTS FOR LOCATION AND SCALE PARAMETERS^{*)}

by

YVES LEPAGE^{**)}

ABSTRACT

A class of nonparametric two-sample tests for testing identity of distributions versus alternatives containing both location and scale parameters is proposed. The asymptotic distribution of the statistics of the class is given under both the hypothesis and a contiguous sequence of alternatives. Some asymptotic optimality properties are deduced for particular tests of the class and finally, the asymptotic efficiency is found.

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1. INTRODUCTION

The purpose of this paper is to propose and study a class of nonparametric two-sample tests for testing the null hypothesis of randomness versus alternatives containing both location and scale parameters. In section 2, the statistics of the class are defined by a quadratic combination of two linear rank statistics.

Section 3 contains the asymptotic distribution of the statistics of the class under both the null hypothesis and a contiguous sequence of alternatives. In section 4, it is proved that the asymptotically maximin most powerful test belongs to the proposed class. Finally, the asymptotic efficiency of the tests of the class relative to the asymptotically maximin most powerful test is given in section 5.

2. THE CLASS OF TESTS

Let $(m_v, m_v), v = 1, 2, \dots$, be a sequence of pairs of positive integers such that $N_v = m_v + m_v \rightarrow \infty$ when $v \rightarrow \infty$. For each v , consider a sequence of random variables X_{v1}, \dots, X_{vN_v} and denote by R_{vi} the rank of X_{vi} among X_{v1}, \dots, X_{vN_v} .

Suppose that under H_v , the random variables X_{v1}, \dots, X_{vN_v} are independently and identically distributed according to a continuous distribution and that under the alternatives, the joint density of X_{v1}, \dots, X_{vN_v} is given by

$$(2.1) \quad q_v = \prod_{i=1}^{m_v} e^{-c_{vi}} f(e^{-c_{vi}} x_i - d_{vi}) \prod_{i=m_v+1}^{N_v} f(x_i)$$

with $c_{vi} = \Delta_1(m_v n_v / N_v)^{-1/2}$ and $d_{vi} = \Delta_2(m_v n_v / N_v)^{-1/2}$, $i=1, \dots, m_v$, where $(\Delta_1, \Delta_2) \in \mathbb{R}^2$ and a known density f in the class C of absolutely continuous density functions on \mathbb{R} such that

$$(2.2) \quad I(f) = \int_0^1 \phi^2(u, f) du < \infty \quad \text{and} \quad I_1(f) = \int_0^1 \phi_1^2(u, f) du < \infty,$$

where if $F(x)$ is the distribution function corresponding to $f(x)$,

$$(2.3) \quad \phi(u, f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))} \text{ and } \phi_1(u, f) = -1 - F^{-1}(u) \frac{f'(F^{-1}(u))}{f(F^{-1}(u))}, \quad 0 < u < 1.$$

Define for $k = 1, 2$, the linear rank statistics

$$(2.4) \quad S_{vk} = \sum_{i=1}^{m_v} a_{vk}(R_{vi}),$$

where $a_{vk}(1), \dots, a_{vk}(N_v)$ are the values of a score function $a_{vk}(\cdot)$. Further, let

$$(2.5) \quad \mu_{vk} = E(S_{vk} | H_v) = \frac{m_v}{N_v} \sum_{i=1}^{N_v} a_{vk}(i) = m_v \bar{a}_{vk}$$

and

$$(2.6) \quad \sigma_{vk}^2 = \text{Var}(S_{vk} | H_v) = \frac{m_v n_v}{N_v(N_v-1)} \sum_{i=1}^{N_v} (a_{vk}(i) - \bar{a}_{vk})^2.$$

Finally, the class of statistics that we will consider is of the form

$$(2.7) \quad U_v = \sum_{k=1}^2 \left(\frac{S_{vk} - \mu_{vk}}{\sigma_{vk}} \right)^2.$$

It should be noted that if $a_{v1}(i) = i$ and $a_{v2}(i) = (N_v+1)/2 - |i - (N_v+1)/2|$, $i=1, \dots, N_v$, the statistics U_v reduce to the combined Wilcoxon and Ansari-Bradley statistics studied by LEPAGE (1971).

3. ASYMPTOTIC DISTRIBUTION

A sequence of score functions $a_v(\cdot)$, $v = 1, 2, \dots$, is said to be generated by a real valued function $\phi(u)$, $0 < u < 1$, if

$$(i) \quad \int_0^1 \phi^2(u) du < \infty \text{ and } \int_0^1 (\phi(u) - \bar{\phi})^2 du > 0 \quad \text{where } \bar{\phi} = \int_0^1 \phi(u) du$$

and,

$$(ii) \quad \lim_{v \rightarrow \infty} \int_0^1 (a_v(1 + [uN_v]) - \phi(u))^2 du = 0 \text{ with } [uN_v] \text{ denoting the largest}$$

integer not exceeding uN_v .

This condition will be needed to obtain the asymptotic distribution under H_v and q_v of the statistics U_v given by (2.7).

Theorem 3.1. Suppose that for $k=1,2$, the sequence of score functions $a_{vk}(\cdot), v = 1,2,\dots$, is generated by $\phi_k(u)$, $0 < u < 1$, with

$$(3.1) \quad \int_0^1 (\phi_2(u) - \bar{\phi}_2)(\phi_2(u) - \bar{\phi}_2) du = 0.$$

Then, under H_v , the statistics U_v are, for $\min(m_v, n_v) \rightarrow \infty$ when $v \rightarrow \infty$, asymptotically χ^2 -distributed with 2 degrees of freedom.

Proof. Let $(\lambda_1, \lambda_2) \in \mathbb{R}^2 - \{(0,0)\}$ and define $T_v = \sum_{i=1}^{m_v} a(R_{vi})$ where for $i = 1, \dots, N_v$, $a_v(i) = \lambda_{v1} a_{v1}(i) + \lambda_{v2} a_{v2}(i)$ with

$$(3.2) \quad \lambda_{vk} = \lambda_k \left(\frac{1}{N_v} \sum_{i=1}^{N_v} (a_{vk}(i) - \bar{a}_{vk})^2 \right)^{-\frac{1}{2}},$$

$k=1,2$. From HÁJEK & ŠIDÁK (1967), p.163, it follows that under H_v , the statistics T_v are asymptotically normal $(0, \lambda_1^2 + \lambda_2^2)$ and consequently, the proof is complete. \square

Theorem 3.2. Suppose that for $k=1,2$, the sequence of score functions $a_{vk}(\cdot), v = 1,2,\dots$, is generated by $\phi_k(u)$, $0 < u < 1$, with relation (3.1) satisfied. Then, under q_v , the statistics U_v are, for $\Delta_1 \neq 0$ and $\min(m_v, n_v) \rightarrow \infty$ when $v \rightarrow \infty$ asymptotically noncentral χ^2 -distributed with 2 degrees of freedom and non-centrality parameter

$$(3.3) \quad \delta^2 = \sum_{k=1}^2 \left[\left(\int_0^1 \phi_k(u) (\Delta_2 \phi(u, f) + \Delta_1 \phi_1(u, f)) du \right)^2 / \int_0^1 (\phi_k(u) - \bar{\phi}_k)^2 du \right].$$

Proof. Consider the statistics T_v defined in the proof of theorem 3.1. From theorem 5.1 of LEPAGE (1973b), one can deduce that under q_v , the statistics T_v are asymptotically normal $(\mu_v, \lambda_1^2 + \lambda_2^2)$ where

$$(3.4) \quad \mu_v = \sum_{k=1}^2 [\lambda_k (\int_0^1 \phi_k(u) (\Delta_2 \phi(u, f) + \Delta_1 \phi_1(u, f)) du)^2 / \int_0^1 (\phi_k(u) - \bar{\phi}_k)^2 du].$$

Consequently, the proof follows. \square

4. ASYMPTOTIC OPTIMALITY

Let C_1 be the class of density functions of C such that

$$(4.1) \quad \int_0^1 \phi(u, f) \phi_1(u, f) du = 0.$$

It should be observed that if $f \in C$ is symmetric with respect to the origin, then $f \in C_1$. Further, define the alternatives $K_v(b)$, $b > 0$, by the joint density q_v given by (2.1) with $f \in C_1$, $\Delta_1 \neq 0$ and

$$(4.2) \quad \Delta_2^2 I(f) + \Delta_1^2 I_1(f) = b^2.$$

In the next theorem, an asymptotically optimal test for H_v versus $K_v(b)$ will be found in the class U_v given by (2.7). The $(1-\alpha)$ -quantile of the χ^2 -distribution with k degrees of freedom will be denoted $\chi_{k, \alpha}^2$ while $F_k(\cdot, \delta^2)$ will represent the distribution function of the non-central χ^2 -distribution with non-centrality parameter δ^2 and k degrees of freedom.

Theorem 4.1. Suppose that $\min(m_v, n_v) \rightarrow \infty$ and that for $k=1, 2$ the sequence of score functions $a_{vk}(\cdot)$, $v = 1, 2, \dots$, are generated by $\phi(u, f)$ and $\phi_1(u, f)$, $0 < u < 1$, respectively. Then, the test based on U_v with critical region

$$(4.3) \quad U_v \geq \chi_{2, \alpha}^2$$

is an asymptotically maximin most powerful test for H_v versus $K_v(b)$ at level α . Furthermore, the asymptotic power is given by

$$(4.4) \quad 1 - F_2(\chi_{2, \alpha}^2, b^2).$$

Proof. The fact that the asymptotic power of the U_v -test equals $1 - F_2(\chi_{2,\alpha}^2, b^2)$ follows immediately from theorem 3.2. Consequently, if $\beta(\alpha, H_v, K_v(b))$ denotes the power of the maximin most powerful test (among all tests), it remains to prove that

$$(4.5) \quad \lim_{v \rightarrow \infty} \beta(\alpha, H_v, K_v(b)) = 1 - F_2(\chi_{2,\alpha}^2, b^2).$$

Suppose that (4.5) does not hold. Thus, passing to a subsequence if necessary, we may assume that

$$(4.6) \quad \lim_{v \rightarrow \infty} \beta(\alpha, H_v, K_v(b)) > 1 - F_2(\chi_{2,\alpha}^2, b^2).$$

By introducing the random variables

$$(4.7) \quad \xi_{v1} = \left(\frac{N_v}{m_v n_v} \right)^{\frac{1}{2}} (I(f))^{-\frac{1}{2}} S_{v1} \quad , \quad \xi_{v2} = \left(\frac{N_v}{m_v n_v} \right)^{\frac{1}{2}} (I_1(f))^{-\frac{1}{2}} S_{v2}$$

and the constants

$$(4.8) \quad \theta_1 = \Delta_2(I(f))^{\frac{1}{2}} \quad , \quad \theta_2 = \Delta_1(I_1(f))^{\frac{1}{2}}$$

one can, in view of theorem 4.1 of LEPAGE (1973b), proceed in a similar way as HÁJEK & ŠIDÁK (1967), p.256-258, and contradict corollary 1 of LEPAGE (1973a). \square

Corollary 4.1. *Under the hypothesis of theorem 4.1, the test based on U_v with critical region (4.3) is an asymptotically uniformly maximin most powerful test for H_v versus $\bigcup_{b>0} K_v(b)$ at level α .*

Proof. In view of relation (4.4) and the definition of uniformly maximin most powerful tests (see HÁJEK & ŠIDÁK (1967), p.29), the result is immediate. \square

Corollary 4.2. *In theorem 4.1 and corollary 4.1, the densities q_v given by (2.1) can be replaced by*

$$(4.9) \quad q'_v = \prod_{i=1}^{m_v} \exp(-\Delta_1 (m_v n_v / N_v)^{-\frac{1}{2}}) f(\exp(-\Delta_1 (m_v n_v / N_v)^{-\frac{1}{2}}) (x_i - \Delta_2 (m_v n_v / N_v)^{-\frac{1}{2}})) \\ \prod_{i=m_v+1}^{N_v} f(x_i).$$

Proof. In view of corollary 5.1 of LEPAGE (1973b), the proof of theorem 4.1 still holds and consequently, corollary 4.1 also. \square

Corollary 4.3. In theorem 4.1 and corollary 4.1, the densities q_v given by (2.1) can be replaced by

$$(4.10) \quad q_{v,\omega} = \prod_{i=1}^{m_v} \exp(-\Delta_1 (m_v n_v / N_v)^{-\frac{1}{2}-\omega}) f(\exp(-\Delta_1 (m_v n_v / N_v)^{-\frac{1}{2}-\omega}) x_i - \Delta_2 (m_v n_v / N_v)^{-\frac{1}{2}}) \\ \prod_{i=m_v+1}^{N_v} e^{-\omega} f(e^{-\omega} x_i)$$

where $\omega \in \mathbb{R}$ is unknown. Furthermore, the test based on U_v with critical region given by (4.3) is then an asymptotically uniformly maximin most powerful α level test for H_v versus $\bigcup_{\omega \in \mathbb{R}} K_v(b)$.

Proof. In view of corollary 5.2 of LEPAGE (1973b), the proof of theorem 4.1 still holds and consequently, the result follows. \square

5. ASYMPTOTIC EFFICIENCY

Let U_v^0 represent the asymptotically maximin most powerful test of theorem 4.1 and consider the tests based on U_v given by (2.7) with critical region $U_v \geq \chi_{2,\alpha}^2$.

Theorem 5.1. Consider testing H_v versus q_v given by (2.1) with $f \in C_1$ and suppose that for $k=1,2$, the sequence of score functions $a_{vk}(\cdot)$, $v = 1,2,\dots$, is generated by $\phi_k(u)$, $0 < u < 1$, with relation (3.1) satisfied. Then, if $\Delta_1 \neq 0$ and $\min(m_v, n_v) \rightarrow \infty$ when $v \rightarrow \infty$, the asymptotic efficiency of U_v relative to U_v^0 , denoted $e(U_v, U_v^0)$, is given by

$$(5.1) \quad e(U_v, U_v^0) = \sum_{k=1}^2 \frac{(\int_0^1 \phi_k(u) (\Delta_2 \phi(u, f) + \Delta_1 \phi_1(u, f)) du)^2}{(\int_0^1 (\phi_k(u) - \bar{\phi}_k)^2 du) (\Delta_2^2 I(f) + \Delta_1^2 I_1(f))}.$$

Proof. It follows from theorem 3.2 that the asymptotic power of the test based on U_v is given by $1 - F_2(\chi_{2, \alpha}^2, \Delta_2^2 I(f) + \Delta_1^2 I_1(f))$. Thus, in view of relation (3.3) and ANDREWS (1954), the proof is complete. \square

It should be noted that if $\phi_1(u) = \phi(u, f_1)$ and $\phi_2(u) = \phi_1(u, f_2)$, $0 < u < 1$, with $f_1, f_2 \in C_1$ and

$$(5.2) \quad \int_0^1 \phi(u, f_1) \phi_1(u, f) du = \int_0^1 \phi(u, f) \phi_1(u, f_2) du = 0,$$

the asymptotic efficiency given by (5.1) can be written as

$$(5.3) \quad e(U_v, U_v^0) = \frac{\Delta_2^2 I(f) e(f_1, f) + \Delta_1^2 I_1(f) e_1(f_2, f)}{\Delta_2^2 I(f) + \Delta_1^2 I_1(f)},$$

where

$$(5.4) \quad e(f_1, f) = \frac{(\int_0^1 \phi(u, f_1) \phi(u, f) du)^2}{I(f_1) I(f)} \text{ and } e_1(f_2, f) = \frac{(\int_0^1 \phi_1(u, f_2) \phi_1(u, f) du)^2}{I_1(f_2) I_1(f)}$$

correspond, respectively to the asymptotic efficiency of the test based on S_{v1} where the sequence of score functions $a_{v1}(\cdot)$, $v = 1, 2, \dots$, is generated by $\phi(u, f_1)$, $0 < u < 1$, relative to the asymptotically most powerful rank test for contiguous location alternatives for a density f and to the asymptotic efficiency of the test based on S_{v2} where the sequence of score functions $a_{v2}(\cdot)$, $v = 1, 2, \dots$, is generated by $\phi_1(u, f_2)$, $0 < u < 1$, relative to the asymptotically most powerful rank test for contiguous scale alternatives for a density f (see HÁJEK & ŠIDÁK (1967), p.267-270).

Furthermore, from (5.3), it can be seen that for all $(\Delta_1, \Delta_2) \in \mathbb{R}^2$,

$$(5.5) \quad \min(e(f_1, f), e_1(f_2, f)) \leq (U_v, U_v^0) \leq \max(e(f_1, f), e_1(f_2, f)).$$

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